HYDRAULIC IMPACT OF "EXPONENTIAL" AND NONLINEARLY VISCOPLASTIC MEDIA IN PIPES MADE OF A VISCOPLASTIC MATERIAL

R. M. Sattarov

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Differential equations are derived and the hydraulic impact process for "exponential" and nonlinearly viscoplastic media in pipes made of a viscoelastic material is analyzed. Hydraulic impact problems for actual media in pipes has been repeatedly treated in the literature [1-4]. The hydraulic impact of a viscous and linearly viscoplastic media in pipes made of an elastic and viscoelastic material was studied in this work. It is well known [5] that many media in the region of low and moderate shear rates reveal a nonlinearity of the flow curve (oil, drilling fluids, polymer solutions and melts, loaded fuels, fuel mixtures, blood, etc.). It should be noted that flexible pipes made of natural materials (pipe boreholes made of polymer materials, membranes of blood vessels, etc.) are described by complicated rheological equations of state for viscoelastic media. Thus a calculation of the influence of nonlinearity of these media and of the viscoelastic properties of the pipe material on the hydraulic impact process is of theoretical and practical interest in many engineering problems.

1. The one-dimensional motion of a droplet of compressible fluid in a pipe of variable cross-section is described by the system of differential equations [1]

$$\frac{\partial M}{\partial t}+\frac{\partial I}{\partial x}=-f\frac{\partial p}{\partial x}-\tau\chi-\gamma f\frac{\partial z}{\partial x}; \tag{1.1}$$

$$\frac{\partial (f\rho)}{\partial t}+\frac{\partial M}{\partial x}=0; \tag{1.2}$$

$$\rho=\rho_0\left(1+\frac{p-p_0}{K_{\text{fl}}}\right),$$

where $M=\int\limits_{(f)}\rho_i v_i df=\rho v f$ is the mass flow rate, $I=\int\limits_{(f)}\rho_i v_i^2 df=(1+\beta)\rho f v^2$ is the projection on the x axis of the momentum of mass M, f is the cross-sectional area of the pipe, v_i and ρ_i are the rate and density of the fluid at a given point, v, ρ, and p are the mean cross-sectional velocity, density, and pressure, z is the height of the center of gravity of the pipe cross sections over the horizontal plane, τ is the tangential stress, χ is the wetted perimeter, γ is the mean specific weight of the fluid, p_0, ρ_0, and f_0 are the values of p, ρ, and f for steady-state motion, and E is the modulus of elasticity of the fluid.

The dependence of the pipe cross-sectional area on time is determined in the following manner.

Denoting the interior radius of a circular pipe by R and the displacement of the radius by u, we obtain

$$f=f_0+2\pi Ru+\pi u^2\approx f_0(1+2u/R).$$

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Since the pipe is thin-walled,

$$\varepsilon = \frac{u}{R}; \quad \sigma\frac{p-p_0}{\delta}R,$$

where σ is stress and δ is the thickness of the pipe walls.

We use the generalized rheological equation of a viscoelastic medium [6]

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i}\sigma = \sum_{i=0}^{n} b_i \frac{d^i}{dt^i}\varepsilon,$$

obtaining the following equation for the law by which f varies with time,

$$\sum_{i=0}^{n} \frac{R}{\delta} a_i \frac{\partial^i (p-p_0)}{\partial t^i} = \sum_{i=0}^{n} \frac{b_i}{2f_0} \frac{\partial^i (f-f_0)}{\partial t^i}. \tag{1.3}$$

The parameters a_i and b_i determine the properties of the material. We will consider below the case when $(p-p_0)/K_{fl} \ll 1$, so that Eqs. (1.1) and (1.2) reduce to the form

$$\frac{\partial M}{\partial t} + \frac{\partial}{\partial x}(1+\beta)\frac{M^2}{\rho_0 f_0} = -\gamma_0 f_0 \frac{\partial}{\partial x}\left(\frac{p-p_0}{\gamma_0} + z\right) - \tau\chi; \tag{1.4}$$

$$\rho_0 \frac{\partial f}{\partial t} + f_0 \frac{\partial \rho}{\partial t} + \frac{\partial M}{\partial x} = 0. \tag{1.5}$$

Bearing in mind the equation $\partial\rho/\partial t = (\rho_0/K_{fl})(\partial p/\partial t)$ and Eq. (1.3), we obtain from Eq. (1.5)

$$\sum_{i=0}^{n} \left[\left(\frac{R}{\delta} a_i + \frac{b_i}{2K_{fl}}\right)\frac{\partial^{i+1} p}{\partial t^{i+1}} + \frac{b_i}{2\rho_0 f_0}\frac{\partial^i}{\partial t^i}\left(\frac{\partial M}{\partial x}\right)\right] = 0. \tag{1.6}$$

The tangential stress in hydraulic impact problems is taken in the form

$$\tau = \frac{C_f}{2}\rho v^2 = \frac{C_f M^2}{2\rho_0 f_0^2},$$

where C_f is the frictional resistance coefficient. If convection terms are ignored in Eq. (1.4), it reduces to the form [1]

$$\frac{1}{f_0}\frac{\partial M}{\partial t} = -\frac{\partial (p - p_0 - \gamma_0 z)}{\partial x} - mM^2, \tag{1.7}$$

where

$$m = \frac{1}{\rho_0 f_0^2}\left[\frac{C_f}{2r} - (1+\beta)\frac{d \ln f_0}{dx}\right]; \tag{1.8}$$

and $r = f_0/\chi$ is the hydraulic radius.

It is well known that the frictional resistance coefficient is inversely proportional to the Reynolds number for laminar motion and, in the case of the motion of an "exponential" fluid [7], has the form

$$C_f = \frac{A}{\mathrm{Re}}\left(\frac{3n+1}{4n}\right) = \frac{A\eta_a}{2\rho_0 vR}\left(\frac{3n+1}{4n}\right), \tag{1.9}$$

and in the case of the motion of a nonlinearly viscoplastic medium [5], the form

$$C_f = \frac{A_1}{\mathrm{Re}'}\left[\left(\frac{4}{3\sigma}\right)^{1/n} + 1\right]^n = \frac{A_1\eta_p}{2\rho_0 Rv}\left[\left(\frac{4}{3\sigma}\right)^{1/n} + 1\right]^n, \tag{1.10}$$

where A and A_i are constant numbers, η_a is the apparent viscosity, η_p is an analog of plastic viscosity, and

$$\beta_0 = \frac{r_0}{R}, \quad \sigma = \frac{1}{\beta_0}\left[1 - \left(\frac{4}{3}\beta_0\right)^{1/n}\right].$$

The constants A and A_1 will be determined for states of nonsteady motion under conditions of nonsteady motion. However, we will henceforth assume, in order to simplify the presentation, that C_f for nonsteady motion is the same Reynolds function as for steady-state motion.

Substituting Eqs. (1.9) and (1.10) in (1.8) and the resulting equation in Eq. (1.7), we obtain

$$\frac{1}{f_0}\frac{\partial M}{\partial t} = -\frac{\partial (p - p_0 + \gamma_0 z)}{dx} - m_j M + qM^2. \tag{1.11}$$

When j = 1, Eq. (1.11) describes the motion of an "exponential" fluid, while when j = 2, Eq. (1.11) describes the motion of a nonlinearly viscoplastic medium,

$$m_1 = \frac{16\eta_a}{4\rho_0 f_0 Rr}\left(\frac{3n+1}{4n}\right);$$

$$m_2 = \frac{16\eta_p}{4\rho_0 f_0 Rr}\left[\left(\frac{4}{3\sigma}\right)^{1/n} + 1\right]^n;$$

$$q = \frac{1+\beta}{\rho_0 f_0^2}\frac{d\ln f_0}{\partial x}.$$

Let us set $P = p - p_0$, so that the differential equations (1.6) and (1.11) are written in the form

$$\begin{cases} \sum_{i=0}^{n}\left[\left(\frac{R}{\delta}a_i + \frac{b_i}{2K\mathrm{fl}}\right)\frac{\partial^{i+1}P}{\partial t^{i+1}} + \frac{b_i}{2\rho_0 f_0}\frac{\partial^i}{\partial t^i}\left(\frac{\partial M}{\partial x}\right)\right] = 0; \\ \frac{1}{f_0}\frac{\partial M}{\partial t} = -\frac{\partial P}{\partial x} - m_j M + qM^2. \end{cases} \tag{1.12}$$

The system of equations (1.12) describes hydraulic impact for the motion of an "exponential" (j = 1) and a nonlinearly viscoplastic (j = 2) medium in pipes made of a viscoelastic material.

For these problems the initial and boundary conditions can be written in the form

$$p(x, 0) = 0; \tag{1.13}$$

$$\frac{\partial p}{\partial t}(x, 0) = 0; \quad \frac{\partial^n p}{\partial t^n} = 0;$$

$$M(x, 0) = 0; \quad p(0, t) = \varphi(t);$$

$$M(l, t) + h\frac{\partial M}{\partial x}(l, t) = F(t),$$

where $\varphi(t)$ and F(t) are given functions and h is a constant.

If the pipe has constant cross section, i.e., if q = 0, the system (1.12) in terms of M(x, t) is written in the form

$$\sum_{i=0}^{n}\left[\frac{b_i}{2\rho_0 f_0}\frac{\partial^i}{\partial t^i}\left(\frac{\partial^2 M}{\partial x^2}\right) - \left(\frac{R}{\delta}a_i + \frac{b_i}{2K\mathrm{fl}}\right)\frac{\partial^{i+1}}{\partial t^{i+1}}\left(\frac{1}{f_0}\frac{\partial M}{\partial t} + m_j M\right)\right] = 0. \tag{1.14}$$

The solution of the differential equation (1.14) under the initial and boundary conditions of Eqs. (1.13) can be carried out by numerical methods or, for example, using Laplace transformations.

We note that the resulting system of differential equations (1.12) reduces in particular cases to hydraulic impact problems well-known in the literature [1-4].

2. Let us consider nonsteady motion of a fluid in a viscoelastic pipe of constant cross section, where the fluid flow rate M is a harmonic function of time of given frequency at the beginning of the pipe; pressure is constant at the end of the pipe. At a moment sufficiently distant from the initial moment, the initial conditions do not practically effect the distribution of flow rate and pressure. Let us find the solution of Eq. (1.14) satisfying the boundary conditions

$$M(0,t) = M_0 e^{i\omega t}; \quad \frac{\partial M}{\partial x}(l,t) = 0. \tag{2.1}$$

The solution of Eq. (1.14) under the boundary conditions of Eqs. (2.1) has the form

$$M(x,\ t) = X_1(x)\cos\omega t + X_2(x)\sin\omega t, \tag{2.2}$$

where $X_1(x) = \mathrm{Re}\{X(x)\}$; $X_2(x) = \mathrm{Im}\{X(x)\}$, and

$$X(x) = \frac{M_0}{\exp(\alpha l) + \exp(-\alpha l)}\{\exp[-\alpha(l-x)] + \exp[\alpha(l-x)]\};$$

$$\alpha^2 = \frac{\sum\limits_{k=0}^{n}\left(\frac{R}{\delta}a_k + \frac{b_k}{2K_{fl}}\right)\left(\frac{1}{f_0}i^{k+2}\omega^{k+2} + i^{k+1}\omega^{k+1}m_j\right)}{\sum\limits_{k=0}^{n} i^k\omega^k\frac{b_k}{2\rho_0 f_0}}.$$

The expressions for $X_1(x)$ and $X_2(x)$ can be explicitly represented if we have an actual value for n. Using the resulting solution (2.2), we may explain the influence of the viscoelastic properties of the pipe material as well as the physical and stress–strain properties of the moving medium on the attenuation of hydraulic impact.

3. To explain the influence of the physical and stress–strain properties of a moving medium as well as that of the viscosity of the material on the attenuation of hydraulic impact we will consider a particular case of the rheological equation for a viscoelastic medium when $a_0 = 1$, $^2/_3[(1+\nu)(1-2\nu)/E]\mu'$; $b_0 = E$, $b_1 = 2\mu'(1+\nu)$; $a_i = b_i = 0$ when $i \geq 2$, where E is the modulus of elasticity, ν is the Poisson coefficient, and μ' is the coefficient of viscosity.

Then the differential equation (2.1) is represented after simple transformations in the form

$$L\frac{\partial^3 M}{\partial t^3} + \frac{\partial^2 M}{\partial t^2}\left(1 + \frac{a}{rf_0}L\right) + m_j f_0\frac{\partial M}{\partial t} = \frac{\partial^2 M}{\partial x^2} + K_2\frac{\partial^3 M}{\partial x^2\partial t}, \tag{3.1}$$

where

$$L = \frac{2\mu'(1+\nu)}{E}\,\frac{1 + \frac{2}{3}\frac{R}{\delta}\frac{(1-2\nu)}{E}K_{fl}}{1 + \frac{2R}{\delta}\frac{K_{fl}}{E}},$$

$$a = \sqrt{\frac{K_1}{\rho_0}}; \quad \frac{1}{K_1} = \frac{1}{K_{fl}} + \frac{2R}{E\delta};$$

$$K_2 = \frac{2\mu'(1+\nu)}{E}.$$

Let us assume that the fluid flow rate M constitutes in some cross section a periodic time function, so that the solution of the differential equation (3.1) can be found in the form

$$M = Ae^{i\omega t + \alpha x}.$$

Substituting the latter equation in Eq. (3.1), we obtain for determining α the equation

$$-L\omega^3 i - \left(1 + \frac{a}{rf_0}L\right)\omega^2 + m_j f_0 i\omega = \alpha^2 + K_2\alpha\lambda i,$$

so that we have

$$\alpha = i\omega\sqrt{\frac{\left(1 + \frac{a}{rf_0}L\right) + Li\omega - m_j f_0\frac{i}{m}}{1 + iK_2\lambda}}.$$

If L, K_2, f_0 and m_j are small, we obtain to an approximation

$$\frac{\alpha}{\omega} = i\left\{1 + i\omega L - \frac{i m_j f_0}{2\omega} - \frac{i K_2 \omega}{2}\right\}. \qquad (3.2)$$

An analysis of the parameter m_1 as a function of n demonstrates that, other conditions being equal, when $n < 1$, m_1 is always greater than its Newtonian analog, and is less when $n > 1$. The value of m_2 increases with increasing β_0 and n relative to the Bingham analog.

It is evident from Eq. (3.2) that: a) attenuation of impact during the motion of an exponential fluid for pseudoplastic media ($n < 1$) occurs significantly more rapidly than for a Newtonian fluid ($n = 1$) and is greater than for dilatant material ($n > 1$):

b) Attenuation of impact occurs more rapidly with the motion of a nonlinearly viscoplastic medium for those media in which the flow limit $\tau_0(\beta_0)$ and nonlinearity parameter n are exceeded.

For an exponential fluid the attenuation coefficient is proportional to

$$\frac{m_1 f_0}{2\omega} = \frac{16\eta_a}{8\rho_0 \omega R r}\left(\frac{3n + 1}{4n}\right),$$

and for a nonlinearly viscoplastic medium, to

$$\frac{m_2 f_0}{2\omega} = \frac{16\eta_p}{8\rho_0 \omega R r}\left[\left(\frac{4}{3\sigma}\right)^{1/n} + 1\right]^n.$$

We may also note that the presence of pipe material viscosity leads to attenuation of impact. In this case the attenuation coefficient will be proportional to

$$(K_2 - L)\,\omega = \omega\,\frac{4\mu'(1 + \theta)^2}{E^2}\,\frac{\frac{2R}{\delta}K_{fl}}{1 + \frac{2R}{\delta}\frac{K_{fl}}{E}}.$$

The results obtained here can be used in solving different concrete problems associated with the drilling of boreholes, the transport of Newtonian media in pipes made of a polymer material, and, in all likelihood, in studying features of blood circulation in the human organism, undergoing the effect of long-term or short-term (impact) G-forces [8].

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